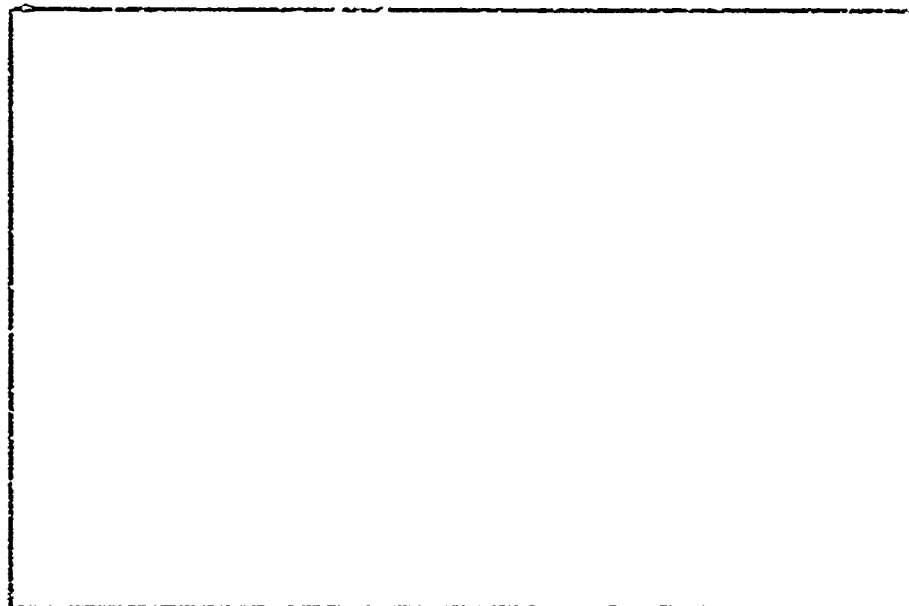# Carnegie-Mellon University
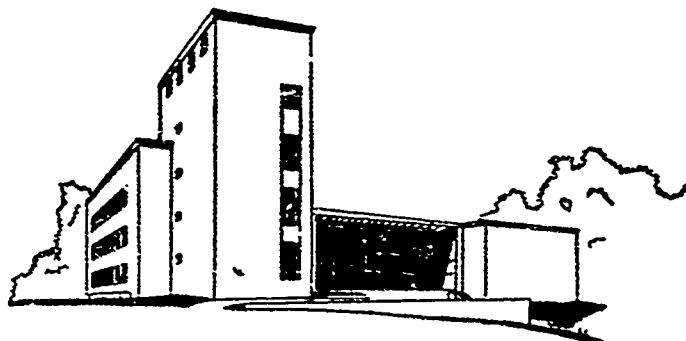
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Management Sciences Research Report No. 284

Polaroids: <u>a new tool in non-convex</u>
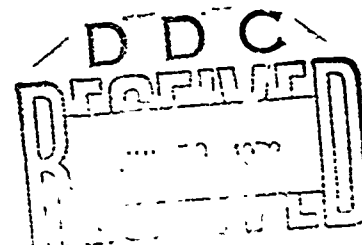<u>and in integer programming</u>

by

Claude-Alain Burdet

February 1972

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13. ABSTRACT

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A list of properties satisfied by polaroids is established indicating that the new concept may be fruitfully used in an area of non-convex (called here polar) programming as well as in integer programming, by means of polaroid cuts; this class of new cuts contains the ones defined by Tuy for concave programming (a special case of polar programming) and by Balas for integer programming; it furthermore provides for new degrees of freedom in the construction of algorithms in the above-mentioned areas of mathematical programming.

II

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
non-linear programming						
integer programming						
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polar sets						
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(quasi)-concave programming						
integer programming						
intersection cuts						

Polaroids: a new tool in non-convex
and in integer programming

by

Claude-Alain Burdet

A B S T R A C T

This paper presents a generalization, called polaroid, of the concept of polar sets.

A list of properties satisfied by polaroids is established indicating that the new concept may be fruitfully used in an area of non-convex (called here polar) programming as well as in integer programming, by means of polaroid cuts; this class of new cuts contains the ones defined by Tuy for concave programming (a special case of polar programming) and by Balas for integer programming; it furthermore provides for new degrees of freedom in the construction of algorithms in the above-mentioned areas of mathematical programming.

$$\underline{\text{IV}}$$

1) Preliminaries

In this introductory section we present the definitions and some relevant properties of a new concept: polaroid sets and functions; these mathematical objects are derived directly from the theory of convex sets and represent a generalization of polar sets; historically, polar relationships have been one of the main topics in projective geometry where the involutory correspondence of poles and polars is of prime interest. The first generalization to polar sets can be found in Minkowski [8]; these sets have since played an increasingly important role in the area of convex analysis and mathematical programming (see for instance, the treatise [10] by Rockafellar); more recently, Balas [2] has uncovered an interesting application of polar sets to integer programming, there the use of a positive-definite quadratic form (n-dimensional sphere or ellipsoid) allows him to define his outer-polar sets which enjoy all the desirable properties for the convex outer-domain theory of valid intersection cuts (further aspects of this theory have been discussed by Glover [5] and Burdet [3,4]).

Our objective here is merely to present some fundamental properties of polaroids, and to indicate how they can be fruitful in an area of non-convex linearly constrained programming (called here polar programming) as well as in integer (primarily zero-one) programming.

When the constrained set is polyhedral, the use of polaroids can be viewed as a generalization of the intersection approach of Hoang Tuy [7] for concave programming or of the intersection cut approach initiated by Balas [1] in integer programming. The generalization with respect to [7] is that polaroid cuts can be defined for non-concave problems;

furthermore, for concave problems Tuy's cuts are <u>uniformly dominated</u> by <u>polaroid cuts;</u> indeed Tuy's approach can be imbedded as a trivial special case in the present theory. With respect to the results of Balas [1,2], the generalization consists in the fact that:

a) convex polaroids can be defined for a more general class

    of functions than positive definite quadratic forms as

    in [2]; adequately constructed polaroids

    can be used as convex outer-domains to generate new

    cutting planes.

b) the convexity requirement imposed on outer-domains

    merely plays the role of a sufficiency condition in the

    construction of an intersection cut,

    and it can be relaxed; it is shown that <u>non-</u>

    <u>convex polaroids</u> may very well be used to generate <u>valid</u>

    <u>polaroid cuts.</u>

<u>Mea culpa</u>: The underlying goal of this report is not primarily focused

            on numerical and computational aspects of the uncovered

properties; the basic idea was to start with as general an object as possible (viz. <u>polaroid sets</u>) and to test those properties which seem promising for the global optimization of non-convex problems. The analysis resulted in a hierarchy of properties ranked by increasingly strong assumptions. In a second effort it was found, however, that many of the additional assumptions leading to the more sophisticated (higher ranked) properties (such as convexity) certainly are convenient because they provide for automatic sufficiency conditions, but at the same time they seem

to be unnecessarily taming the power of the approach. For instance, for
the construction of a cutting plane it is believed (as indicated in the con-
clusions) that a fruitful line of research (in order to attain computational
efficiency) would be to proceed along the following way:

a) generate a cut under very weak a priori assumptions (from
an arbitrary polaroid, for instance).

b) check a posteriori the validity of the cut (using theorems
13 or 14); this may require an ad hoc weakening

readjustment of the cutting plane but will in general yield

a better cut than if sufficiency conditions had been postu-

lated a priori for the entire polaroid.

2) Polaroids

Let $f = f(x,y)^{1/}$ be a real valued function with two n-vector argu-
ments x and y ; let P denote a closed set in $\mathbb{R}^n$.

Definition 1: For a given value of the parameter k , define the polaroid
set $P*(k)$ by

$$P*(k) = \{y \mid f(x,y) \leq k , \forall x \epsilon P\} \tag{1}$$

By convention set $P*(k) = \emptyset$, $\forall k < \min_{y} \{f(x,y) \mid x \epsilon P\}$

Theorem 1 (Inclusion theorem):

For any closed sets P and $Q \subset \mathbb{R}^n$, one has the following implica-
tions:

If $Q \subset P$ then $Q*(k) \supset P*(k)$; and

If $k_1 \geq k_2$ then $P*(k_1) \supset P*(k_2)$.

Proof: The assertions follow from (1):

$$Q*(k) = \{y \mid f(x,y) \le k , \forall x \epsilon Q\} \supset \{y \mid f(x,y) \le k , \forall x \epsilon P \supset Q\} = P*(k)$$

$$P*(k_1) = \{y \mid f(x,y) \le k_1, \forall x \epsilon P\} \supset \{y \mid f(x,y) \le k_2 \le k_1 , \forall x \epsilon P\} = P*(k_2)$$

Q.E.D.

Theorem 2 (Union theorem): For any closed sets P , Q one has

$$(P \cup Q)* (k) = P*(k) \cap Q*(k)$$

Proof: $(P \cup Q)* (k) = \{y \mid f(x,y) \le k , \forall x \epsilon (P \cup Q)\} =$

$$= \{y \mid f(x,y) \le k , \forall x \epsilon P\} \cap \{y \mid f(x,y) \le k , \forall x \epsilon Q\}$$

$$= P*(k) \cap Q*(k) .$$

Q.E.D.

Corollary 2.1: Let

$$(P \triangle Q) = cl \{x \mid x \epsilon (P \cup Q) \text{ and } x \notin (P \cap Q)\}$$

Then one has

$$(P \cap Q)^* (k) \cap (P \triangle Q)^* (k) = P^*(k) \cap Q^*(k)$$

Proof: $(P \cup Q)^* (k) = [(P \cup Q) \cup (P \triangle Q)]^*(k) = (P \cap Q)^*(k) \cap (P \triangle Q)^*(k)$

On the other hand, theorem 2 implies:

$$(P \cup Q)^*(k) = P^*(k) \cap Q^*(k)$$

Q.E.D.

Theorem 3 (Intersection theorem):

$$(P \cap Q)^* (k) \supset P^*(k) \cup Q^*(k)$$

Proof: $(P \cap Q)^* (k) = \{y \mid f(x,y) \le k , \forall x \epsilon (P \cap Q)\}$

$$\supset \{y \mid f(x,y) \le k , \forall x \epsilon P\} \cup \{y \mid f(x,y) \le k , \forall x \epsilon Q\} =$$

$$= P^*(k) \cup Q^*(k) .$$

Q.E.D.

Corollary 2.2:

$$P^*(k) = (P \cap Q)^*(k) \cap (P \Delta (P \cap Q))^*(k)$$

$$Q^*(k) = (P \cap Q)^*(k) \cap (Q \Delta (P \cap Q))^*(k)$$

Proof: $P = (P \cap Q) \cup (P \Delta (P \cap Q))$, and similarly for Q ; applying

the union theorem completes the proof. Q.E.D.

Corollary 2.3:

$$P^*(k) \cup Q^*(k) = (P \cap Q)^*(k) \cap [(P \Delta (P \cap Q))^*(k) \cup (Q \Delta (P \cap Q))^*(k)]$$

$$\supset (P \cap Q)^*(k) \cap (P \Delta Q)^*(k) = P^*(k) \cap Q^*(k)$$

Proof: The (first) equality is obtained directly from the corollary 2.2;

the inclusion can be derived by inspection of the polaroid sets

in the bracket or directly from corollary 2.1, since $P^* \cup Q^* \supset P^* \cap Q^*$.

Definition 2: The function $g = g(x) = f(x,x)$ is said <u>polarized</u> by f .

Denote the <u>level set</u> of g by $\operatorname{lev}_\alpha g = \{x \mid g(x) \leq \alpha\}$.

Definition 3: An arbitrary (closed compact) set S satisfying

$$\emptyset \neq S \cap P \subset \operatorname{lev}_k g$$

is called a <u>valid cut at the level</u> k .

Theorem 4: A cut S, valid at the level k_1 , is valid at all higher

levels $k_2 (\geq k_1)$.

Proof: Immediate since $\operatorname{lev}_{k_1} g \subset \operatorname{lev}_{k_2} g$ by definition.

Q.E.D.

3) Convex Polaroids

Theorem 5: The polaroid $P*(k)$ is <u>convex</u> $\forall k$

        <u>iff</u> $f(x,y)$ is <u>quasi-convex in</u> y, for all $x \varepsilon P$.

Proof: Let y^1 and y^2 be arbitrary points in $P*(\cdot)$; and set

$$y^3 = \lambda y^1 + (1-\lambda)y^2 \ , \ 0 \leq \lambda \leq 1$$

then $\forall x \varepsilon P$, one has

$$f(x,y^3) = f(x,\lambda y^1 + (1-\lambda)y^2) \leq \max \{f(x,y^1) \ , \ f(x,y^2)\} = k$$

Furthermore, assume $P*(k)$ convex for all k ; thus let $y^3 = \lambda y^1 + (1-\lambda)y^2$, then one has $y^3 \varepsilon P*(k)$, for all k such that y^1 and $y^2 \varepsilon P*(k)$ i.e. for all k such that:

$$k \geq \max \{f(x,y^1) \ , \ f(x,y^2) \mid x \varepsilon P\} \ .$$

In particular for

$$k = \bar{k} \max \{f(x,y^1) \ , \ f(x,y^2) \mid x \varepsilon P\} \ .$$

Since $y^3 \varepsilon P*(\bar{k})$ one has $\forall x \varepsilon P$:

$$f(x,y^3) \leq \bar{k} = \max \{f(x,y^1) \ , \ f(x,y^2)\} \ , \ \forall x \varepsilon P$$

<div align="right">Q.E.D.</div>

In order to acquire at this point a better geometrical feeling for the content of the statement in theorem 5 let us review some classical results:

1) Let f be the <u>euclidean scalar product</u> $f(x,y) = \sum\limits_{i=1}^{n} x_i y_i = \langle x,y \rangle$

    which yields (the square of) the <u>euclidean norm as polarized</u> <u>function</u> $g(x) = \sum\limits_{i=1}^{n} x_i^2$.

Following the classical definition [10] one finds

that, in this case, the polaroid P*(1) is nothing but the

polar set P*

$$P*(1) = P* = \{x* \mid \langle x, x* \rangle \leq 1 , \forall x \in P\}$$

In [2] Balas introduced a generalization of the classical

concept by allowing for a scaling factor k in the

above definition. Clearly this can be absorbed in our

definition, either by considering the polaroid $P^*(k)$

with respect to the same polaroid function f , or by

changing the polaroid function to $\tilde{f} = \frac{1}{k} f$ and retaining

the parameter 1 : $\tilde{P}^*$ (1) .

Since f is bilinear it is quasi-convex in y and theorem 5

paraphrases, in this case, the convex property of

polar sets.

2) Let f be the scalar product function corresponding to a

general Riemann metric, with arbitrary (real symmetric)

metric tensor g_{ik} ; i.e.

$$f(x,y) = \sum_{i,k=1}^{n} g_{ik} x_i y_k ,$$

for the polarized function g one has

$$g(x) = \sum_{i,k=1}^{n} g_{ik} x_i x_k$$

Since the parameter k can always be absorbed by the tensor

g_{ik} , the polaroid $P* = P*(1)$ is a genuine generalization

of the polar concept, to arbitrary quadratic forms (not

necessarily definite or positive).

Since f is again bilinear, one finds from theorem 5

that the polaroid P* generated from an arbitrary set P

is convex; in a follow-up paper, we show how the polaroid

theory can be used to solve an arbitrary non-convex quadratic programming problem (particularly in the indefinite case) [12].

3) The following extension of a Riemann metric was found useful to generate convex outer-domains in integer programming [4].

Let $f(x,y) = \sum_{i=1}^{n} g_i x_i y_i$

where $g_i = \begin{cases} = \Delta_i (a_i^+)^{-2} & \text{for } x_i > 0 \\ = 0 & \text{for } a_i^+ = 0 \text{ or } a_i^- = 0 \\ = \Delta_i (a_i^-)^{-2} & \text{for } x_i < 0 \end{cases}$

with $\sum_{i=1}^{n} \Delta_i = 1$, $\Delta_i \geq 0$, (a_i^+ and a_i^- being given quantities).

The corresponding polarized function g

$$g(x) = \sum_{i=1}^{n} \varepsilon_i x_i^2 \quad \text{is}$$

piecewise quadratic, i.e. it is quadratic in each of the 2^n "orthants of $\mathbb{R}^n$." But here again because $f(x,y)$ is linear in y for every x in a given orthant, the polaroid set $P^*(k)$ can be shown to be convex.

4) Consider a quasi-convex function $g(x)$ and define

$$f(x,y) = g(y) \quad , \quad \forall x$$

The polaroid $P*(k)$ then merely reduces to $P*(k) = \text{lev}_k\, g$ and theorem 5 restates the known convexity property of $\text{lev}_k\, g$. In general, however, there belongs many other possibilities for choosing $f = f(x,y)$ such that $f(x,x) = g(x)$;

hence there will correspond other polaroid sets to

the same polarized function g(x) ; example 1 illus-

trates this situation, for instance.

5) This list could extend indefinitely! As a final example

consider

$$f = f(x,y) = \sum_{i,j=1}^{n} (c_{ij} + \sum_{k=1}^{n} d_{ijk}x_k) \, y_i y_j$$

and suppose that P has the property that the matrix

[C + Dx] is <u>positive semi-definite</u> ∀x∈P . Then

f(x,y) is convex ∀x∈P and the polaroid P* is

<u>convex.</u>

4) <u>Complete polaroids</u>

Until now we always considered k as an accessory parameter whose

value played no essential role. We now analyze <u>polaroids</u> (convex or not)

corresponding to <u>particular values</u> of k .

<u>Definition 4</u>: The polaroid P*(k) defined by (1) is called <u>complete</u> if

$$P \subset P*(k)$$

<u>Definition 5</u>: A function f = f(x,y) is called <u>complete on P</u> at the level k if

$$f(x,y) \le k \quad , \quad \forall x,y \in P$$

<u>Theorem 6</u> (<u>Completeness theorem</u>):

The polaroid P*(k) is complete <u>iff</u> f is complete on P (at the level k).

<u>Proof</u>: Take any x'∈P ; then f(x,x') ≤ k , ∀x∈P by hypothesis, when

f is complete; but one has $P*(k) = \{y \mid (f(x,y)) \le k , \forall x \in P\}$,

which shows that x' must belong to P*(k) and hence P ⊆ P*(k) .

Conversely, suppose P*(k) complete; the argument is by contradiction:

suppose there exists a pair x,y∈P such that f(x,y) > k ; in this

case one has either $x \notin P*(k)$ or $y \notin P*(k)$, or both; in any case

$P \not\subset P*(k)$ which is contrary to hypothesis.

<div align="right">Q.E.D.</div>

The following theorems will be useful in establishing

optimality conditions for polar programming as well as for testing the

validity of polaroid cuts in integer programming.

<u>Corollary 6.1</u>: If $P^*(k)$ is complete then $P \subset \text{lev}_k g$.

<u>Proof</u>: By hypothesis one has $f(x,y) \leq k$, $\forall x, y \in P$; hence, in par-

ticular for $y = x$: $f(x,x) = g(x) \leq k$.

<div align="right">Q.E.D.</div>

<u>Theorem 7</u>: If f is symmetric, i.e. $f(x,y) = f(y,x)$ then

$$P \subset (P^*(k))^*(k)$$

<u>Proof</u>: $P^*(k) = \{y \mid f(x,y) \leq k$, $\forall x \in P\}$, hence

$f(x,y) \leq k$, $\forall x \in P$, $\forall y \in P^*(k)$; but by hypothesis

$f(x,y) = f(y,x) \leq k$, hence $\forall x \in P$, one has

$x \in (P^*(k))^*(k) = \{y \mid f(y,u) \leq k$, $\forall y \in P^*(k)\}$

and therefore $P \subset (P^*(k)^*(k)$
<div align="right">Q.E.D.</div>

<u>Corollary 7.1</u>: If $P^*(k)$ is complete (and f symmetric) then

$$P \subset (P^*(k))^*(k) \subset P^*(k)$$

<u>Proof</u>: $P \subset P^*(k)$ implies

$(P^*(k))^*(k) = \{y \mid f(x,y) \leq k, \forall x \in P^*(k)\} \subset$

$\subset \{y \mid f(x,y) \leq k, \forall x \in P\} = P^*(k)$

<div align="right">Q.E.D.</div>

Let us now consider the "boundary" sets

$$\text{bd } P*(k) = \{y \epsilon P*(k) \mid f(x,y) = k \quad, \quad \text{for some} \quad x \epsilon P\}$$

$$\text{bd } (\text{lev}_k g) = \{x \mid g(x) = k\} \ .$$

Theorem 8: Suppose $P*(k)$ is complete, then one has

$$[P \cap \text{bd}(\text{lev}_k g)] \subseteq [P \cap \text{bd } P*(k)]$$

Proof: First let us note that the assertion is trivial for

$$[P \cap \text{bd}(\text{lev}_k g)] = \emptyset \ .$$

Take $x \epsilon [P \cap \text{bd}(\text{lev}_k g)]$, i.e. $x \epsilon P$ with $g(x) = k$. Since $P*(k)$ is complete, one also has $x \epsilon P*(k)$. Thus $x \epsilon P*(k)$ and $g(x) = f(x,x) = k$, with $x \epsilon P$ which completes the proof.

<div align="right">Q.E.D.</div>

Theorem 6 indicates that when $P*(k)$ is complete, k is an upper bound for the polarized function g on the set P . Theorem 8 now states that this upper bound may only be attained on the "boundary" of the polaroid $P*(k)$. Thus completeness means that no interior point of $P*(\bar{k})$ is an optimal polar program $\bar{x} \epsilon P$, with

$$\bar{k} = \max_{x \epsilon P} g(x) = g(\bar{x}) \ .$$

This is stated in the following

Corollary 8.1 (Boundary Theorem).

If $P*(\bar{k})$ is complete, then every optimal solution $\bar{x}$ of the polar programming problem

$$\text{maximize } g(x) \quad, \quad \text{subject to} \quad x \epsilon P$$

satisfies $\bar{x} \epsilon \text{bd } P*(\bar{k})$.

Proof: Optimality implies $\bar{x} \epsilon \text{bd lev}_k g$; theorem 8 completes the proof.

<div align="right">Q.E.D.</div>

The above corollary, in a way, corresponds to the classical result asserting that the maximum of a quasi-convex function over a closed (bounded) set P is always attained on the boundary of P.

Theorem 9: Let $Q \subset P$; if $P*(k)$ is complete, then $Q*(k)$ is complete.

Proof: From the inclusion theorem 1 one has $P*(k) \subset Q*(k)$ and completeness of $P*(k)$ yields

$$Q \subset P \subset P*(k) \subset Q*(k)$$

Q.E.D.

Theorem 10: Let $Q \subset P$ and $P*(k)$ be complete; then one has

$$Q \subset P \subset \text{lev}_k g$$

and

$$(\text{lev}_k g)* \ (k) \subset P*(k) \subset Q*(k)$$

Proof: Immediate from theorems 1 and 7.

Q.E.D.

Corollary 10.1: If $P*(k)$ and $(\text{lev}_k g)* \ (k)$ are complete, then one has

$$Q \subset P \subset \text{lev}_k g \subset (\text{lev}_k g)* \ (k) \subset P*(k) \subset Q*(k)$$

Proof: Immediate from theorem 10.

Q.E.D.

Theorem 11: If $(P \cup Q)* \ (k)$ is complete, then both $P*(k)$ and $Q*(k)$ are complete.

Proof: From the union theorem one has

$$(P \cup Q) \subset (P \cup Q)* \ (k) = P*(k) \cap Q*(k) \ ;$$

but $P \subset (P \cup Q)$ so that $P \subset P*(k) \cap Q*(k) \ ;$

and hence $P \subset P*(k)$; and similarly for Q .

Q.E.D.

In integer programming, one is really interested in the polaroid set of isolated sets (points in all integer-, or linear fibers in mixed integer-programming). Completeness of the outer-domain is clearly a desired feature in order to generate a deep cut and theorem 11 indicates that such an outer domain can be constructed as the intersection of individual polaroids (one for each point or fiber) provided all such polaroids contain all feasible integer points.

The same argument holds true, of course, in polar programming if the feasible set P consists of (or is arbitrarily split into) several components.

5) Validity and optimality conditions

The definition 3 of a valid cut allows one to consider optimality conditions in polar programming as a particular valid cut; this is formulated in the next

Theorem 12: The polarized function $g(x) = f(x,x)$ attains its maximal value $\bar{k} = g(\bar{x})$ over the set P at the point $\bar{x} \in P$

iff there exists a valid cut S at the level $\bar{k} = \max\limits_{x \in P} g(x) = g(\bar{x})$

Proof: Trivial (take for instance $S = \text{lev}_{\bar{k}} g$)   Q.E.D.

In this section we are not only interested in stating necessary and sufficient optimality conditions for polar programs in terms of more general (and computationally more easily tractable) polaroids than the (trivial) level sets lev g ; we also want to find the conditions which must be satisfied, for an arbitrary cut to be valid; for instance, the term valid cut can here be visualized as stemming from the cutting plane approach of Tuy [7] for concave programming, or from the intersection cut approach, in integer programming, [1,4,5]. In tne latter case, however, a somewhat stronger concept for valid cuts is necessary (see section 7).

__Theorem 13__: If the set $(P \cap S)*(k)$ is complete then

S is a valid cut at the level k .

__Proof__: Immediate from theorem 7, applied to the set $(P \cap S)$.

__Theorem 14__: Every subset S of a polaroid $P*(k)$ is a valid cut.

__Proof__: (Eliminate the uninteresting case where $S \cap P = \emptyset$.) One has

$$P*(k) = \{y \mid f(x,y) \le k \ , \ \forall x \in P\}$$

hence $\quad f(x,y) \le k \quad \forall x \in P \ , \ \forall y \in S \subset P*(k)$ and in particular

$$f(x,y) \le k \ , \ \forall x,y \in P \cap S$$

Thus the function f is complete on the set $(P \cap S)$; theorems 6 and

7 establish that the definition 3 is satisfied. __Q.E.D.__

__Corollary 14.1__: $P \cap P^*(k) \subset \text{lev}_k g$

__Proof__: 1) One may simply set $S = P^*(k)$ and apply Theorem 14 together with Definition 3.

       2) Alternately, the proof can be obtained as follows:

$$P^*(k) = \{y \mid f(x,y) \le k, \ \forall x \in P\}$$

hence $y \in P^*(k) \Rightarrow f(x,y) \le k, \ \forall x \in P$ and, in particular,

when $x = y \in P$, one has

$$f(y,y) = g(y) \le k, \ \forall y \in P \cap P^*(k)$$

__Q.E.D.__

     In theorems 13 and 14 one notes that neither of them requires

completeness of $P*(k)$; all that is required in theorem 13 is that the

cutoff portion of P (i.e. $(P \cap S)$) be contained in the polaroid

$(P \cap S)*(k)$; from the intersection theorem 3 one has

$$(P \cap S)*(k) \supset P*(k) \cup S*(k)$$

showing that the hypothesis of theorem 13 is quite weak and should

be easy to test computationally; in particular when one constructs S as

a subset of $P*(k)$ (as is often practically the case) theorem 14

shows that one merely must ensure $S \subset P*$.

<u>Corollary 6.2</u>: (<u>Optimality theorem</u>)

The point $\bar{x}$ is an optimal solution of the <u>polar programming</u>

problem

$$\text{maximize } g(x) \text{ , subject to } x \epsilon P$$

<u>iff</u> there exists a complete polaroid $P*(k)$ such that

$$\bar{x} \epsilon \text{ bd } P*(k)$$

<u>Proof</u>: Immediate from theorems 7 and 8.   Q.E.D.

6) <u>Polaroid cuts for linearly constrained polar or integer programming problems.</u>

Let us now focus our attention on the <u>polyhedral</u> sets $P \subset \mathbb{R}^n$ i.e.

$$P = \{x = x_1,\dots,x_n) \mid x_i = \bar{x}_i - \sum_{j\epsilon\bar{N}} \bar{a}_{ij} t_j \geq 0 \text{ , } \forall i\epsilon(N \cup M) \tag{2}$$

where it is assumed that (2) represents a linear program in explicit

(Tucker) format, expressed with respect to a basis with non-basic set

$\bar{N} \subset (N \cup M)$; $N = \{1.2,\dots,n\}$ is the set of the <u>original variables</u>;

$M = \{n+1,\dots,n+m\}$ is the set of the <u>slack (and artificial) variables</u>.

Consider the extreme (basic) ray $u^j(t_j) = \bar{x} - \bar{a}_j t_j$, $t_j \geq 0$, $j\epsilon\bar{N}$

where $\bar{a}_j = (\bar{a}_{1j},\bar{a}_{2j},\dots,\bar{a}_{nj})$,

$\bar{x} = (\bar{x}_1,\bar{x}_2,\dots,\bar{x}_n)$,

and assume (for simplicity of the exposition) that $\bar{x} \epsilon \text{ Int } P*(k)$,

i.e. that the intersection points $u^j(\bar{\lambda}^j)$ of each ray u^j, $j\epsilon\bar{N}$ with

the "boundary" of $P*(k)$ are different from $\bar{x}$, that is

$$u^j(\bar{\lambda}^j) = \bar{x} - \bar{\lambda}^j\bar{a}_j \text{ , } \text{ with } \bar{\lambda}^j > 0$$

Furthermore define the halfspaces

$$H^+ = \{x = x(t) \mid \sum_{j \in \bar{N}} \frac{1}{\bar{\lambda}^j} \, t_j > 1\}$$

$$H^- = \{x = x(t) \mid \sum_{j \in \bar{N}} \frac{1}{\bar{\lambda}^j} \, t_j < 1\}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is given by the n first components of the linear program (2) characterizing P. Clearly H^+ and H^- are <u>open</u>, and they are both defined by the <u>cutting plane</u>

$$\sum_{j \in \bar{N}} \frac{1}{\bar{\lambda}^j} \, t_j = 1$$

(Note that, (by assumption) $\bar{x} \in H^-$, i.e. $\bar{x} \notin H^+$)

Define the set $S \subset P^*(k)$ as the following n-dimensional simplicial hull
$$S = conv \{\bar{x}, \, u^j(\bar{\lambda}^j) \,, \quad \forall j \in \bar{N}\}$$

Assuming that the hypothesis of one of the theorem 13 or 14 is satisfied the set S represents a <u>valid cut</u>. (See section 7) Implementation into the current linear programming tableau merely amounts to the addition of the <u>new linear constraint</u>

$$\sum_{j \in \bar{N}} \frac{1}{\bar{\lambda}^j} \, t_j \geq 1$$

called <u>polaroid cut</u>.

For the use of a cutting plane method to solve non-convex problems the reader is referred to the papers by Hoang Tuy [7], Glover [5] and Klingman [11], Gomory [6], Balas [1,2] or Burdet [3,13]. A more detailed study of particular types of polaroid cuts in integer programming is given in Burdet [4,12].

7) Conclusions

In polar programming the objective function is the polarized function g ; algorithms for solving such non-convex optimization problems therefore follow a stepwise construction of a set S satisfying

$$P \subset S \subset P*(\bar{\bar{k}})$$

where $\bar{\bar{k}}$ is always the current best value and ultimately $\bar{k} = \max\limits_{x \in P} g(x)$. The optimality theorem provides for the stopping criterion. In practice, the use of polaroids will prove efficient whenever the set $P*(\bar{k})$ is much larger than P ; in this case ample room is left for an easy construction of a set S yielding a sufficiency condition for global optimality.

In integer programming, the use of polaroids is somewhat different since they are only used for the characterization of the set of feasible integer solutions, as compared to the other (continuous) feasible solutions; the polarized function g then usually has the property:

$$g(v) = 1 \qquad \forall v = \text{vertex of a unit cube } U(\bar{x})$$
$$\text{containing the linear programming}$$
$$\text{optimum } \bar{x} .$$

$$g(x) \leq 1 \qquad \forall x \in U(\bar{x})$$

thus $\bar{k} = 1$; the meaning of a valid cut S here becomes: Int(S) contains no feasible integer point (as shown by theorem 13, the construction of a valid cut requires neither completeness nor convex-

ity of the polaroid P^*). However, a stronger definition of valid cuts

is needed here; we need add the requirement

$$\text{Int}(S) \cap [P \cap \text{bd lev}_k g] = \emptyset$$

to the definition 3.

In terms of this new definition, Theorem 13 remains true if one requires

the additional hypothesis:

$S \subset (P \cap S)^*(k)$; indeed, by theorem 8:

$$(P \cap S \cap \text{bd lev}_k g) \subset (P \cap S \cap \text{bd } (P \cap S)^*(k))$$

every feasible integer point which lies in S also lies on the boundary of S :

$$\text{bd } S \supset (P \cap S \cap \text{bd lev}_k g)$$

(Proof : since $\text{Int}(S) \cap \text{bd } (P \cap S)^*(k) = \emptyset$ one has

$\text{Int}(S) \cap \text{bd lev}_k g \subset \emptyset$, Q.E.D.)

This general result opens a new area of research which contrasts from the

algebraic approach of Gomory [6] or the intersection approach of Balas [1]

where one imposes in advance conditions which are sufficient to ensure the

validity of the cut independently of the particular simplex S which

is actually generated. The polaroid cut approach also entails such

a possibility (using convex and complete polaroids); but, in addition,

it can lead to new methods where an arbitrary polaroid P^* is used to

generate the simplex S (owing to the generality of P^* , this cut

can be made deep); in a second phase, the validity of S is then

established; for instance by checking directly $S \subset P^*$ (theorem 14).

Footnotes

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[1]/ f is in fact a <u>bifunction</u> [10] with object functions f(x;·), for x∈P; f can also be viewed as a <u>multivalued</u> <u>mapping</u> $\mathbb{R}^n \to \mathbb{R}$, where each x∈P determines a particular mapping.

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